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Topological algebraic structure on Souslin and Aronszajn lines

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ABSTRACT

Z. Feng and R.W. Heath proved that any separable linearly ordered space (LOTS) which is a cancellative topological semigroup must be metrizable. In this note, we show that the same holds more generally for CCC LOTS by proving that no Souslin line admits a continuous cancellative binary operation. We also show that no Lindelöf Aronszajn line admits such an operation.

1. Introduction

A *topological semigroup* is a triple $(X,\tau,*)$ such that (X,τ) is a topological space and the binary operation $*: X\times X\to X$ is continuous and associative. A topological semigroup is *cancellative* if $a*b=a*c$ or $b*a=c*a$ implies $b=c$. If X is in fact a group (i.e., an identity element and inverses also exist), then $(X,\tau,*)$ is called a *paratopological group*, and if the inverse operation is also continuous, it is a *topological group*.

Clearly, any topological group is a paratopological group, and since the inverse operation exists in a paratopological group (though need not be continuous), any paratopological group is a cancellative topological semigroup. Any space X can be a topological semigroup in a trivial way, e.g., fix $x_0\in X$ and let $xy=x_0$ for all $x,y\in X$. But it is reasonable to ask what spaces, in particular what linearly ordered spaces (LOTS), can be cancellative topological semigroups or something stronger.

A nonmetrizable LOTS can be a topological group: an old example due to Dieudonne (see [7]) is $\mathbb{R}^{\omega_1}$ with the lexicographic order ($lex(\mathbb{R}^{\omega_1})$) under coordinatewise addition. This example is not first countable, and of course cannot be since first countable topological groups are metrizable (see [8] or [2]).

So we ask: what type of group structure, weaker than a topological group, can there be on a nonmetrizable first countable LOTS? Since any first countable paratopological group must have a G_δ-diagonal [3], and an LOTS with a G_δ-diagonal is metrizable [9], a first countable nonmetrizable LOTS cannot be a paratopological group. This leads us to consider the question of the existence of a first countable nonmetrizable LOTS which supports a cancellative topological semigroup structure. Feng and Heath [5] showed that any connected or separable LOTS which is a cancellative topological semigroup must be metrizable. It is natural to ask whether "separable" can be weakened to "CCC" in this result. We show that the answer is positive by showing that no Souslin line, which we define to be a CCC nonseparable LOTS, admits a continuous cancellative binary operation.

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This leaves open the question whether any first countable nonmetrizable LOTS can be a cancellative topological semigroup. We don't answer this question, but we are able to show that no Lindelöf Aronszajn line admits a continuous cancellative binary operation.

We remark that there is a difference here between LOTS and GO-spaces (subspaces of LOTS). The Sorgenfrey line is a first countable nonmetrizable GO-space which is a paratopological group under usual addition of real numbers.

2. Souslin lines and Aronszajn lines

Before embarking on the proof of our main result, that no Souslin line admits a continuous cancellative binary operation, we give the idea, which came out of an attempt to embed a Souslin line in $lex(\mathbb{R}^{\omega_1})$ and to use the operation of coordinatewise addition, which is continuous on $lex(\mathbb{R}^{\omega_1})$.[1]

Suppose X is a subset of $\mathbb{R}^{\omega_1}$ with the following properties:

(a) X with the lexicographic order contains a nontrivial convergent sequence;
(b) Each $x \in X$ ends in a string of 0's;
(c) For any $\alpha < \omega_1$, there are two points $x, y \in X$ such that $x \upharpoonright \alpha = y \upharpoonright \alpha$;
(d) X is closed under coordinatewise addition.

We claim that coordinatewise addition on X is not continuous. To see why, choose a sequence a_n, $n \in \omega$, of distinct points in X converging to some point $a \in X$, say from the left. Let $\alpha < \omega_1$ be such that all the a_n's and a are 0 at all coordinates $\geqslant \alpha$. Choose distinct points x and y such that $x \upharpoonright \alpha = y \upharpoonright \alpha$; without loss of generality, $x < y$. Since $a_n < a$ and a_n and a disagree below α and are 0 greater than or equal to α, we have $a_n + y < a + y$, and the first coordinate of disagreement of $a_n + y$ and $a + y$ is below α. Now $a + x$ and $a + y$ agree below α since x and y do, so $a_n + y < a + x$. Thus $a_n + y < a + x < a + y$ for all n, so $a_n + y$ cannot converge to $a + y$. Thus addition isn't continuous.

The idea of the proof given below is to show that something similar to the above occurs in any Souslin line with a supposed continuous cancellative binary operation.

Theorem 1. *No Souslin line admits a continuous cancellative binary operation.*

Proof. Suppose X is a Souslin line with a continuous cancellative binary operation $*$. We will denote $a * b$ by ab.

We define in a standard way a collection $T = \bigcup_{\alpha<\omega_1} \mathcal{I}_\alpha$ of open convex subsets of X which is a Souslin tree under reverse inclusion. Start by letting $D_0 = \emptyset$, $I_0 = X$ and $\mathcal{I}_0 = \{I_0\}$. If D_α and $\mathcal{I}_\alpha$ have been defined, then given $I \in \mathcal{I}_\alpha$, choose a nonempty closed discrete subset $D(I)$ which is both cofinal and coinitial in I, and let

$$D_{\alpha+1} = D_\alpha \cup \bigcup \{D(I)\colon\ I \in \mathcal{I}_\alpha\}$$

and let $\mathcal{I}_{\alpha+1}$ be the set of convex components of $X \setminus D_{\alpha+1}$. If α is a limit ordinal and D_β and $\mathcal{I}_\beta$ have been defined for $\beta < \alpha$, let

$$D_\alpha = \overline{\bigcup_{\beta<\alpha} D_\beta}$$

and let $\mathcal{I}_\alpha$ be the convex components of $X \setminus D_\alpha$. Note the following key facts about this construction:

(1) Each D_α is separable and each $D_{\alpha+1} \setminus D_\alpha$ is countable;
(2) If $I_0 \in \mathcal{I}_\alpha$ and $I_1 \in \mathcal{I}_\beta$ with $\alpha < \beta$, and $I_0 \cap I_1 \neq \emptyset$, then $\overline{I_1} \subset I_0$;
(3) $X = \bigcup_{\alpha<\omega_1} D_\alpha$.

The argument is standard: (1) is an easy induction and (2) is clear from the construction. For (3), suppose $p \in X \setminus \bigcup_{\alpha<\omega_1} D_\alpha$. Then for each $\alpha < \omega_1$, $p \in I_\alpha$ for some $I_\alpha \in \mathcal{I}_\alpha$. Let J_α be a convex component of $I_\alpha \setminus D_{\alpha+1}$ not containing p. Then the J_α's are pairwise-disjoint, contradiction.

By (3), for each $x \in X$ there is a least ordinal α_x with $x \notin \bigcup \mathcal{I}_{\alpha_x}$ (or equivalently, $x \in D_{\alpha_x}$); define $h : X \to \omega_1$ by $h(x) = \alpha_x$.

Claim 1. *For each $\alpha < \omega_1$, the set*

$$\{(y, z)\colon\ y, z \in D_{\alpha+1} \cup \{\pm\infty\}\}$$

is a base for the points of D_α.

[1] As we mentioned, $\mathbb{R}^{\omega_1}$ is not first countable, but subsets thereof with the restricted order topology, which may be different from the subspace topology, can be.

Proof. Let $p \in D_\alpha$. The claim clearly holds if $D_{\alpha+1} \cap (\leftarrow, p)$ is cofinal in $(\leftarrow, p)$ and $D_{\alpha+1} \cap (p, \rightarrow)$ is coinitial in $(p, \rightarrow)$. The proofs are similar, so we only show $D_{\alpha+1} \cap (\leftarrow, p)$ is cofinal in $(\leftarrow, p)$.

If $D_\alpha \cap (\leftarrow, p)$ is cofinal in $(\leftarrow, p)$, we are done, so suppose not. Then the set J of points q such that $q < p$ and $q > x$ for any $x \in D_\alpha \cap (\leftarrow, p)$ is a convex component of $X \setminus D_\alpha$, hence $J \in \mathcal{I}_\alpha$. Then $D(J)$ is cofinal in J, hence in $(\leftarrow, p)$, and $D(J) \subset D_{\alpha+1}$. $\square$

Claim 2. *Suppose $y, z \in D_\alpha$, $y < z$, and $I \in \mathcal{I}_\beta$, $\beta \geqslant \alpha$. Then either $I \cap (y, z) = \emptyset$ or $I \subset (y, z)$.*

Proof. This follows easily since I is convex and $I \cap D_\alpha = \emptyset$. $\square$

Let us call two points x and y in X *adjacent* if one is the immediate successor of the other.

Claim 3. *Suppose $Y \subset X$ and $f : Y \to X$ is one-to-one and continuous. If $\{h(y):\ y \in Y\}$ is unbounded in ω_1, so is $\{h(f(y)):\ y \in Y\}$.*

Proof. Suppose $f : Y \to X$ witnesses otherwise. Then $f(Y) \subset D_\alpha$ for some $\alpha < \omega_1$. Let D be a countable dense subset of $D_{\alpha+1}$. Since Y is uncountable and f is one-to-one, by passing to an uncountable subset of Y if necessary we may assume no two points of $f(Y)$ are adjacent. Then by Claim 1, there is a point of D between any two points of $f(Y)$, so the collection

$$\mathcal{U} = \big\{(\leftarrow, d):\ d \in D\big\}$$

is a countable collection of open sets separating the points of $f(Y)$ in the T_0-sense.

Then $\{f^{-1}(U):\ U \in \mathcal{U}\}$ T_0-separates Y. For each $U \in \mathcal{U}$ and $y \in f^{-1}(U)$, by Claim 1 we can choose a basic open set $B(y, U)$ with $y \in B(y, U) \subset f^{-1}(U)$, where $B(y, U)$ is an interval with endpoints in $D_{h(y)+1} \cup \{\pm\infty\}$. Since X is hereditarily Lindelöf, there is a countable subcollection $\mathcal{B}(U)$ of $\{B(y, U):\ y \in f^{-1}(U) \cap Y\}$ covering its union. Let $\mathcal{B} = \bigcup\{\mathcal{B}(U):\ U \in \mathcal{U}\}$. Then $\mathcal{B}$ also T_0-separates Y.

Let $\delta \in \omega_1$ be strictly greater than $h(y)$ for every y with $B(y, U) \in \mathcal{B}$. If $y \in Y$ and $h(y) > \delta$, there is some $I_y \in \mathcal{I}_\delta$ with $y \in I_y$. There are $y \neq z \in Y$ with $I_y = I_z = I$. By Claim 2, for each $B \in \mathcal{B}$, either $I \cap B = \emptyset$ or $I \subset B$. But then no member of $\mathcal{B}$ separates y and z, contradiction. $\square$

Claim 4. *For each $a \in X$, there is a club $C \subset \omega_1$ such that, for each $\alpha \in C$, if $h(x) \geqslant \alpha$, then $h(ax) \geqslant \alpha$.*

Proof. Suppose Claim 4 fails for $a \in X$. Then there is a stationary set S and a point $x_\alpha \in X$ for each $\alpha \in S$ such that $h(x_\alpha) \geqslant \alpha$ but $h(ax_\alpha) = \beta_\alpha < \alpha$. By the Pressing Down Lemma, there is an ordinal β and an uncountable $A \subset \omega_1$ such that $\beta_\alpha = \beta$ for every $\alpha \in A$. Let $Y = \{x_\alpha:\ \alpha \in A\}$. The map $f : Y \to X$ defined by $f(y) = ay$ is one-to-one and continuous, but $\{h(y):\ y \in Y\}$ is unbounded while $\{h(f(y)):\ y \in Y\}$ is bounded, contradicting Claim 3. $\square$

Claim 5. *Let $a \neq b \in X$. Then there is $\alpha < \omega_1$ such that, for every $y \in X$, ay and by are not in the same member of $\mathcal{I}_\alpha$ (and hence not in the same member of $\mathcal{I}_\beta$ for any $\beta \geqslant \alpha$).*

Proof. Suppose otherwise. Let $y_\alpha \in X$ be such that $ay_\alpha, by_\alpha \in I_\alpha \in \mathcal{I}_\alpha$; we can assume, without loss of generality, that $ay_\alpha < by_\alpha$. Either ay_α and by_α are adjacent and so $(\leftarrow, ay_\alpha]$ and $[by_\alpha, \rightarrow)$ are clopen, or there is δ_α and $d_\alpha \in D_{\delta_\alpha}$ with $ay_\alpha < d_\alpha < by_\alpha$. By continuity of $*$, there is a basic open set B_α containing y_α such that, either (i) $aB_\alpha \subset (\leftarrow, ay_\alpha]$ and $bB_\alpha \subset [by_\alpha, \rightarrow)$, or (ii) $aB_\alpha \subset (\leftarrow, d_\alpha)$ and $bB_\alpha \subset (d_\alpha, \rightarrow)$.

There is a countable set C such that $\{B_\alpha:\ \alpha \in C\}$ covers $\bigcup_{\alpha \in \omega_1} B_\alpha$. Choose $\beta \in \omega_1$ greater than $sup(\{h(ay_\alpha) + \delta_\alpha:\ \alpha \in C\})$; note that $[ay_\beta, by_\beta] \subset I_\beta$. Now $y_\beta \in B_\alpha$ for some $\alpha \in C$. If we are in the situation (i) of the previous paragraph, then $ay_\beta \leqslant ay_\alpha < by_\alpha \leqslant by_\beta$, which puts ay_α in I_β. But $\beta > h(ay_\alpha)$, contradiction. On the other hand, if we are in situation (ii), then $ay_\beta < d_\alpha < by_\beta$, so $I_\beta \cap D_{\delta_\alpha} \neq \emptyset$, a contradiction since $\delta_\alpha < \beta$. $\square$

The following claim is the contradiction which completes the proof of the theorem.

Claim 6. *The operation $*$ cannot be continuous.*

Proof. Suppose $*$ were continuous. Let $a_n \to a$ in X, where the a_n's and a are all distinct. For each $c \neq d \in X$, let $\alpha(c, d) \in \omega_1$ be as in Claim 5. Let $\delta \in \omega_1$ such that

$$\delta > sup\big\{\alpha(a_n, a):\ n \in \omega\big\}.$$

By Claim 4, there is $y \in X$ such that $h(y) > \delta$ and $h(ay) > \delta$. Let $I \in \mathcal{I}_\delta$ be such that $ay \in I$. Then by continuity of $*$, $a_n y \in I$ for all sufficiently large n. But $\delta > \alpha(a_n, a)$ implies $a_n y$ and ay are not in the same member of $\mathcal{I}_\delta$, contradiction. $\square$

Remark. Note that only separate continuity of the supposed continuous binary operation was used in the above proof.

The following corollary now follows immediately from Feng and Heath's result [5] that a separable LOTS which is a cancellative topological semigroup is metrizable.

Corollary 2. *A CCC LOTS which is a cancellative topological semigroup is metrizable.*

Additionally, the remark stated above leads to a corollary for semitopological groups, and we thank the referee for this insight. A semitopological group G is a group with topology such that the binary operation $*: G \times G \to G$ is separately continuous. Obviously, a semitopological group need not be a cancellative topological semigroup.

Corollary 3. *A CCC LOTS which is a semitopological group is metrizable.*

Proof. By the theorem, such an LOTS is separable, and, hence, first countable. Then by Proposition 3.6 in [1], the space has a G_δ-*diagonal*, and, therefore, it is metrizable. $\square$

In considering the more general question whether a first countable nonmetrizable LOTS can be a cancellative topological semigroup, it is natural to consider Aronszajn lines. An Aronszajn line is a linear ordering of cardinality $\aleph_1$ containing no subset that's order isomorphic to ω_1 (with the usual ordering), the reverse of ω_1, or an uncountable subset of $\mathbb{R}$.

Funk and Lutzer [6] show that an Aronszajn line is hereditarily paracompact and zero-dimensional, but it is neither compact nor separable. In particular, any Aronszajn line satisfying the CCC is a Souslin line.

Other than the properties mentioned above, the topology of Aronszajn lines can vary greatly; such lines can even be metrizable. E.g., Funk and Lutzer note that for any Aronszajn line, X, the lexicographic product $Y = X \times \mathbb{Z}$ with the open interval topology is both a discrete metric space and an Aronszajn line; hence it will of course support a topological group operation. We don't know if an Aronszajn line which is a cancellative topological semigroup must be metrizable, but we can adapt our Souslin line argument to show that a Lindelöf Aronszajn line cannot admit a continuous cancellative binary operation. We'll use the fact (see [10]) that every Aronszajn line can be realized as an Aronszajn tree (i.e., a tree of height $\aleph_1$ with every level and branch countable) with a lexicographic order topology. Recall that a lexicographic order on a tree T is defined as follows. First assign a linear order to each node, where a *node* is a maximal collection of elements of T all having exactly the same set of predecessors. Then the corresponding lexicographic order $\prec$ is defined by $s \prec t$ iff s is less than t in the tree order, or $s(\alpha)$ is less than $t(\alpha)$ in the node order, where α is least such that $s(\alpha) \neq t(\alpha)$ (where for $u \in T$, $u(\alpha)$ denotes the predecessor of u at level α).

Theorem 4. *No Lindelöf Aronszajn line admits a continuous cancellative binary operation.*

Proof. Let $(T, <_T)$ be an Aronszajn tree with a lexicographic order $\prec$. Suppose T with this lexicographic order is Lindelöf and T admits a continuous cancellative binary operation. We will denote the product of s and t under this operation by st. Let $h(t)$ denote the height of t in T. For $s \neq t$, let $\Delta(s,t)$ be $min\{h(s), h(t)\}$ if $s \leqslant_T t$ or $t \leqslant_T s$, and let it be $min\{\alpha\colon\ s(\alpha) \neq t(\alpha)\}$ otherwise.

Claim. *Let $a \neq b \in T$. Then there is $\alpha(a,b) < \omega_1$ such that, for every $y \in T$, $\Delta(ay, by) \leqslant \alpha(a,b)$.*

Proof. Suppose not. Let $y_\alpha \in T$ such that $\Delta(ay_\alpha, by_\alpha) \geqslant \alpha$. Since T is Lindelöf, there is a point $y \in T$ such that, for every neighborhood N of y, the set $\{\alpha\colon\ y_\alpha \in N\}$ is uncountable. Then $ay \neq by$. There are two cases to consider, and we will obtain a contradiction in each case.

Case 1. $ay < by$ or $by < ay$. Assume, without loss of generality, $ay < by$. There are $\alpha_n \in \omega_1$ with each $\alpha_n > h(by)$ such that $y_{\alpha_n} \to y$. Since $ay_{\alpha_n} \to ay \prec by$, we have that $ay_{\alpha_n} \ngeq by$ for all sufficiently large n. Since also $by_{\alpha_n} \to by$, it follows that $\Delta(ay_{\alpha_n}, by_{\alpha_n}) < h(by)$ for all sufficiently large n, contradicting $\Delta(ay_{\alpha_n}, by_{\alpha_n}) \geqslant \alpha_n > h(by)$.

Case 2. There is δ such that $ay(\delta) \neq by(\delta)$. Again choose $\alpha_n > \delta$ such that $y_{\alpha_n} \to y$. Since $ay_{\alpha_n} \to ay$, we have $ay_{\alpha_n}(\delta) = ay(\delta)$ for sufficiently large n. Similarly, $by_{\alpha_n}(\delta) = by(\delta)$ for sufficiently large n. Hence $\Delta(ay_{\alpha_n}, by_{\alpha_n}) \leqslant \delta$ for sufficiently large n, contradiction. Thus the claim is proved. $\square$

To finish the proof of the theorem, choose $a_n, a \in T$ with $a_n \to a$. Let $\delta > sup\{\alpha(a_n, a)\colon\ n \in \omega\}$. Since the map $y \mapsto ay$ is one-to-one, we can choose $y \neq y' \in T$ with $h(ay), h(ay') > \delta$ and $ay(\delta) = ay'(\delta)$. Without loss of generality, $ay \prec ay'$. Then $ay(\delta) \prec ay \prec ay'$, so for all sufficiently large n, we have $ay(\delta) \prec a_n y \prec ay'$, which implies $\Delta(a_n y, ay) > \delta$ for such n, contradiction. $\square$

Question 1. Can any nonmetrizable Aronszajn line be a cancellative topological semigroup?

For this question it might help to know how to tell from the corresponding Aronszajn tree and the node orderings whether or not a given Aronszajn line is metrizable, or Lindelöf, or ... But such questions seem to be open as well; see, e.g., final remarks in Funk and Lutzer [6].

Of course, the following is the main question left open by this paper:

Question 2. Is there a nonmetrizable first countable LOTS which is a cancellative topological semigroup?

We remark that, if a first countable LOTS X is a counterexample to Question 2, translation cannot be surjective (i.e., it cannot be that $aX = X$ and $Xa = X$ for all $a \in X$): a semigroup with surjective translations is a group (see, e.g., [4, p. 6]), so X would be a paratopological group and hence metrizable.

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